

# Dubrovin's duality for $F$ -manifolds with eventual identities

Liana David and Ian A.B. Strachan

**Abstract:** A vector field  $\mathcal{E}$  on an  $F$ -manifold  $(M, \circ, e)$  is an eventual identity if it is invertible and the multiplication  $X * Y := X \circ Y \circ \mathcal{E}^{-1}$  defines a new  $F$ -manifold structure on  $M$ . We give a characterization of such eventual identities, this being a problem raised by Manin [12]. We develop a duality between  $F$ -manifolds with eventual identities and we show that is compatible with the local irreducible decomposition of  $F$ -manifolds and preserves the class of Riemannian  $F$ -manifolds. We find necessary and sufficient conditions on the eventual identity which insure that harmonic Higgs bundles and  $DChk$ -structures are preserved by our duality. We use eventual identities to construct compatible pair of metrics.

## 1 Introduction

In [4] Dubrovin introduced the idea of an almost dual Frobenius manifold. Starting from a Frobenius manifold one may construct a new geometric object that shares many, but crucially not all, of the essential features of the original manifold. In particular a new 'dual' solution of the underlying Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations may be constructed from the original manifold. Such a construction reflects certain other 'dualities' that occur in other areas of mathematics where Frobenius manifolds appear. For example, in:

- Quantum cohomology and mirror symmetry;
- Integrable systems, via generalizations of the classical Miura transform;
- Singularity theory, via the correspondence between oscillatory integrals and period integrals.

More specifically, given a Frobenius manifold  $(M, \circ, e, E, \tilde{g})$  with multiplication  $\circ$ , unity field  $e$ , Euler field  $E$  and metric  $\tilde{g}$  one may define a new multiplication  $*$  and metric  $g$  by the formulae

$$\begin{aligned} X * Y &= X \circ Y \circ E^{-1}, \\ g(X, Y) &= \tilde{g}(E^{-1} \circ X, Y) \end{aligned}$$

where  $E^{-1} \circ E = e$ . Clearly  $*$  is associative, commutative and has a unity, namely  $E$ , the original Euler field. The new metric  $g$  (the intersection form) turns out to be flat and from these two new objects one may define a dual solution to the WDVV-equations. This correspondence is not completely dual

- certain properties are lost. For example, while  $\widetilde{\nabla}e = 0$ , the new identity does not share this property: in general  $\nabla E \neq 0$ .

Underlying Frobenius manifolds is a structure known as an  $F$ -manifold, which was introduced by Hertling and Manin [8].

**Definition 1.** [8] *i) An  $F$ -manifold is a triple  $(M, \circ, e)$  where  $M$  is a manifold,  $\circ$  is a commutative, associative multiplication on the tangent bundle  $TM$ , with identity vector field  $e$ , such that the  $F$ -manifold condition*

$$L_{X \circ Y}(\circ) := X \circ L_Y(\circ) + Y \circ L_X(\circ), \quad (1)$$

*holds, for any smooth vector fields  $X, Y \in \mathcal{X}(M)$ .*

*ii) An Euler vector field (of weight  $d$ ) on an  $F$ -manifold  $(M, \circ, e)$  is a vector field  $E$  which preserves the multiplication up to a constant, i.e.*

$$L_E(\circ)(X, Y) = d X \circ Y, \quad \forall X, Y \in \mathcal{X}(M).$$

$F$ -manifolds appear in many areas of mathematics. All Frobenius manifolds have an underlying  $F$ -manifold structure, and in examples originating from singularity theory such  $F$ -manifolds arise in a very natural way [7]. They also appear within integrable systems - both in examples coming from the submanifold geometry of Frobenius manifolds [16] and non-local biHamiltonian geometry [2] and their role has been elucidated further in [10].

Given an  $F$ -manifold with an Euler vector field one may construct a dual multiplication via  $X * Y = X \circ Y \circ E^{-1}$ . While this is commutative and associative with unity element, whether or not this defines an  $F$ -manifold is not immediately clear. More generally, Manin [12] replaced the Euler field  $E$  by an arbitrary invertible vector field and used this to define a new multiplication.

**Definition 2.** [12] *A vector field  $\mathcal{E}$  on an  $F$ -manifold  $(M, \circ, e)$  is called an eventual identity, if it is invertible (i.e. there is a vector field  $\mathcal{E}^{-1}$  such that  $\mathcal{E} \circ \mathcal{E}^{-1} = \mathcal{E}^{-1} \circ \mathcal{E} = e$ ) and, moreover, the multiplication*

$$X * Y = X \circ Y \circ \mathcal{E}^{-1}, \quad \forall X, Y \in \mathcal{X}(M) \quad (2)$$

*defines a new  $F$ -manifold structure on  $M$ .*

The reason for the terminology is that  $\mathcal{E}$  is the identity vector field for the multiplication  $*$ . In this paper we give the characterization of such eventual identities, thus answering a question raised by Manin [12].

**Theorem 3.** *i) Let  $(M, \circ, e)$  be an  $F$ -manifold and  $\mathcal{E}$  an invertible vector field. Then  $\mathcal{E}$  is an eventual identity if and only if*

$$L_{\mathcal{E}}(\circ)(X, Y) = [e, \mathcal{E}] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M). \quad (3)$$

*ii) Let*

$$X * Y = X \circ Y \circ \mathcal{E}^{-1}$$

*be the new  $F$ -manifold multiplication. Then the map*

$$(M, \circ, e, \mathcal{E}) \rightarrow (M, *, \mathcal{E}, e)$$

*is an isomorphism between  $F$ -manifolds with eventual identities.*

Condition (3) above may be seen as a generalization of the notion of an Euler vector field. All invertible Euler vector fields are eventual identities but not conversely. However, eventual identities play a similar role. In this paper we study  $F$ -manifolds with eventual identities and their relation with some well-known constructions in the theory of Frobenius manifolds.

The plan of the paper is the following. In Section 2 we prove Theorem 3 and we develop its consequences. We remark that the duality for  $F$ -manifolds with eventual identities developed in Theorem 3 *ii*) is a natural generalization of the well-known dualities for almost Frobenius manifolds and for  $F$ -manifolds with compatible flat structures [4], [12]. After proving Theorem 3 we show that any eventual identity on a product  $F$ -manifold is a sum of eventual identities on the factors (a similar decomposition holds for Euler vector fields [7]). Using this fact we show that our duality for  $F$ -manifolds with eventual identities is compatible with the local irreducible decomposition of  $F$ -manifolds [7]. We end Section 2 with examples and further properties of eventual identities, some of them being already known for Euler vector fields.

In Section 3 we add a new ingredient on our  $F$ -manifold  $(M, \circ, e, \mathcal{E})$  with eventual identity, namely a multiplication invariant metric  $\tilde{g}$ . The eventual identity  $\mathcal{E}$  together with  $\tilde{g}$  determine, in a canonical way, a second metric  $g$ , defined like the second metric of a Frobenius manifold. We prove that the metrics  $(g, \tilde{g})$  are almost compatible. Our main result in this Section states that  $(g, \tilde{g})$  are compatible, when  $(M, \circ, e, \tilde{g})$  is an almost Riemannian  $F$ -manifold, i.e. the coidentity  $\epsilon \in \Omega^1(M)$ , which is the 1-form dual to the identity  $e$ , is closed. Similar results already appear in the literature [2], with Euler vector fields instead of eventual identities.

In Section 4 we show that our duality for  $F$ -manifolds with eventual identities preserves the class of Riemannian  $F$ -manifolds, which are almost Riemannian  $F$ -manifolds satisfying an additional curvature condition. Riemannian  $F$ -manifolds were introduced and studied in [10] and are closely related to the theory of integrable systems of hydrodynamic type.

In Section 5 we apply our results to the theory of integrable systems.

In Section 6 we study the interactions between  $tt^*$ -geometry, introduced for the first time in [1], and our duality of  $F$ -manifolds with eventual identities.  $tt^*$ -geometry shares many properties in common with Frobenius manifolds, its main ingredients being a metric, a Higgs field and a real structure (the latter not being present in the theory of Frobenius manifolds). One can combine  $tt^*$ -geometry with Frobenius manifold theory giving rise to new structures (like CDV-structures,  $DChk$ -structures, etc) satisfying some complicated compatibility conditions, but which are very natural in examples coming from singularity theory. It is in this context that  $F$ -manifolds appear in  $tt^*$ -geometry. We determine necessary and sufficient conditions on the eventual identity which insure that the class of harmonic Higgs bundles and  $DChk$ -structures (i.e. harmonic Higgs bundles with compatible real structure) is preserved by our duality for  $F$ -manifolds with eventual identities.

## 2 Eventual identities and duality

In this Section we prove Theorem 3. We begin with a simple preliminary Lemma concerning invertible vector fields on  $F$ -manifolds.

**Lemma 4.** Let  $(M, \circ, e)$  be an  $F$ -manifold and  $\mathcal{E}$  an invertible vector field, with inverse  $\mathcal{E}^{-1}$ . Assume that

$$L_{\mathcal{E}}(\circ)(X, Y) = [e, \mathcal{E}] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M). \quad (4)$$

Then also

$$L_{\mathcal{E}^{-1}}(\circ)(X, Y) = [e, \mathcal{E}^{-1}] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M). \quad (5)$$

*Proof.* The proof is a simple calculation. Since  $e = e \circ e$ , the  $F$ -manifold condition (1) with  $X = Y := e$  implies that  $L_e(\circ) = 0$ . Applying again (1) with  $X := \mathcal{E}$  and  $Y := \mathcal{E}^{-1}$ , we obtain:

$$0 = L_{\mathcal{E} \circ \mathcal{E}^{-1}}(\circ) = \mathcal{E} \circ L_{\mathcal{E}^{-1}}(\circ) + \mathcal{E}^{-1} \circ L_{\mathcal{E}}(\circ).$$

Combining this relation with (4) we get

$$L_{\mathcal{E}^{-1}}(\circ)(X, Y) = \mathcal{E}^{-2} \circ [\mathcal{E}, e] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M),$$

where  $\mathcal{E}^{-2}$  denotes  $\mathcal{E}^{-1} \circ \mathcal{E}^{-1}$ . On the other hand,

$$[e, \mathcal{E}] \circ \mathcal{E}^{-2} = (L_e(\mathcal{E}) \circ \mathcal{E}^{-1}) \circ \mathcal{E}^{-1} = (L_e(e) - \mathcal{E} \circ L_e(\mathcal{E}^{-1})) \circ \mathcal{E}^{-1} = [\mathcal{E}^{-1}, e]$$

where we used  $L_e(\circ) = 0$ . Our claim follows.  $\square$

Note that the construction of  $\mathcal{E}^{-1}$ , whilst just linear algebra, requires the inversion of a matrix, and hence  $\mathcal{E}^{-1}$  is not defined at points of  $M$  where a certain determinant  $\Sigma$  vanishes. Rather than defining a new manifold  $M^* \cong M \setminus \Sigma$  on which  $\mathcal{E}^{-1}$  is defined we just assume that  $M$  consists of points at which both  $\mathcal{E}$  and  $\mathcal{E}^{-1}$  are well defined.

After this preliminary result, we now prove Theorem 3 stated in the Introduction.

**Proof of Theorem 3.** The multiplication  $*$  is commutative, associative, with identity field  $\mathcal{E}$ . Therefore  $(M, *, \mathcal{E})$  is an  $F$ -manifold if and only if for any vector fields  $Z, V \in \mathcal{X}(M)$ ,

$$L_{Z*V}(\circ)(X, Y) = Z * L_V(\circ)(X, Y) + V * L_Z(\circ)(X, Y), \quad \forall X, Y \in \mathcal{X}(M). \quad (6)$$

We will show that (6) is equivalent with (3). For this, we take the Lie derivative with respect to  $Z$  of the relation (2). We get, by a straightforward computation,

$$L_Z(\circ)(X, Y) = L_Z(\circ)(\mathcal{E}^{-1} \circ X, Y) + L_Z(\circ)(\mathcal{E}^{-1}, X) \circ Y + [Z, \mathcal{E}^{-1}] \circ X \circ Y. \quad (7)$$

Using relation (7) with  $Z$  replaced by  $Z * V = Z \circ V \circ \mathcal{E}^{-1}$  and the  $F$ -manifold condition (1) satisfied by the multiplication  $\circ$ , we get:

$$\begin{aligned} L_{Z*V}(\circ)(X, Y) &= \mathcal{E}^{-1} \circ Z \circ L_V(\circ)(\mathcal{E}^{-1} \circ X, Y) + \mathcal{E}^{-1} \circ V \circ L_Z(\circ)(\mathcal{E}^{-1} \circ X, Y) \\ &\quad + Z \circ V \circ L_{\mathcal{E}^{-1}}(\circ)(\mathcal{E}^{-1} \circ X, Y) + \mathcal{E}^{-1} \circ Z \circ Y \circ L_V(\circ)(\mathcal{E}^{-1}, X) \\ &\quad + \mathcal{E}^{-1} \circ V \circ Y \circ L_Z(\circ)(\mathcal{E}^{-1}, X) + Z \circ V \circ Y \circ L_{\mathcal{E}^{-1}}(\circ)(\mathcal{E}^{-1}, X) \\ &\quad - L_{\mathcal{E}^{-1}}(\mathcal{E}^{-1} \circ Z \circ V) \circ X \circ Y. \end{aligned}$$

Combining this expression with the expressions of  $L_Z(*) (X, Y)$  and  $L_V(*) (X, Y)$  provided by (7), we see that (6) holds if and only if

$$\begin{aligned} X \circ Y \circ (L_{\mathcal{E}^{-1}}(\mathcal{E}^{-1} \circ Z \circ V) + \mathcal{E}^{-1} \circ Z \circ [V, \mathcal{E}^{-1}] + \mathcal{E}^{-1} \circ V \circ [Z, \mathcal{E}^{-1}]) = \\ Z \circ V \circ (L_{\mathcal{E}^{-1}}(\circ)(\mathcal{E}^{-1} \circ X, Y) + Y \circ L_{\mathcal{E}^{-1}}(\circ)(\mathcal{E}^{-1}, X)). \end{aligned}$$

On the other hand, it can be checked that

$$\begin{aligned} L_{\mathcal{E}^{-1}}(\mathcal{E}^{-1} \circ Z \circ V) + \mathcal{E}^{-1} \circ Z \circ [V, \mathcal{E}^{-1}] + \mathcal{E}^{-1} \circ V \circ [Z, \mathcal{E}^{-1}] \\ = L_{\mathcal{E}^{-1}}(\circ)(\mathcal{E}^{-1}, Z) \circ V + L_{\mathcal{E}^{-1}}(\circ)(\mathcal{E}^{-1} \circ Z, V). \end{aligned}$$

Hence  $*$  is the multiplication of an  $F$ -manifold structure if and only if for any vector fields  $X, Y, Z, V \in \mathcal{X}(M)$ ,

$$\begin{aligned} X \circ Y \circ (L_{\mathcal{E}^{-1}}(\circ)(\mathcal{E}^{-1} \circ Z, V) + L_{\mathcal{E}^{-1}}(\circ)(\mathcal{E}^{-1}, Z) \circ V) \\ = Z \circ V \circ (L_{\mathcal{E}^{-1}}(\circ)(\mathcal{E}^{-1} \circ X, Y) + L_{\mathcal{E}^{-1}}(\circ)(\mathcal{E}^{-1}, X) \circ Y). \end{aligned}$$

Taking  $X = Y := e$  it is easy to see that this relation is equivalent with

$$L_{\mathcal{E}^{-1}}(\circ)(\mathcal{E}^{-1} \circ Z, V) + L_{\mathcal{E}^{-1}}(\circ)(\mathcal{E}^{-1}, Z) \circ V = -2\mathcal{E}^{-1} \circ [\mathcal{E}^{-1}, e] \circ Z \circ V. \quad (8)$$

We now simplify relation (8). For this, we take in (8)  $Z := e$  and we obtain

$$L_{\mathcal{E}^{-1}}(\circ)(\mathcal{E}^{-1}, V) = -\mathcal{E}^{-1} \circ [\mathcal{E}^{-1}, e] \circ V, \quad \forall V \in \mathcal{X}(M). \quad (9)$$

Combining (8) with (9) we get:

$$L_{\mathcal{E}^{-1}}(\circ)(Z, V) = -[\mathcal{E}^{-1}, e] \circ Z \circ V, \quad \forall Z, V \in \mathcal{X}(M). \quad (10)$$

Conversely, it is clear that if (10) is satisfied then (8) is satisfied as well. Therefore, relations (8) and (10) are equivalent. We proved that  $*$  is the multiplication of an  $F$ -manifold structure if and only if (10) holds. Our first claim follows from Lemma 4.

For our second claim, assume that  $\mathcal{E}$  is an eventual identity on an  $F$ -manifold  $(M, \circ, e)$ . We want to prove that  $e$  is an eventual identity for the  $F$ -manifold  $(M, *, \mathcal{E})$ , where  $*$  is related to  $\circ$  by (2). Since the identity field of  $*$  is  $\mathcal{E}$ , we need to show that

$$L_e(*) (X, Y) = [\mathcal{E}, e] * X * Y, \quad \forall X, Y \in \mathcal{X}(M). \quad (11)$$

Letting  $Z := e$  in (7) and using  $L_e(\circ) = 0$  together with (2), we get:

$$L_e(*) (X, Y) = [e, \mathcal{E}^{-1}] \circ X \circ Y = ([e, \mathcal{E}^{-1}] \circ \mathcal{E}^2) * X * Y.$$

Recall now from the proof of Lemma 4 that  $[e, \mathcal{E}^{-1}] \circ \mathcal{E}^2 = [\mathcal{E}, e]$ . Our second claim follows. The proof of Theorem 3 is now completed.

Having found the characterization of eventual identities one may study how such objects may be combined to form new eventual identities.

**Proposition 5.** *i) Eventual identities form a subgroup of the group of invertible vector fields on an  $F$ -manifold.*

*ii) The Lie bracket of two eventual identities is an eventual identity, provided that is invertible.*

*iii) Let  $(M_1 \times M_2, \circ, e_1 + e_2)$  be the product of two  $F$ -manifolds  $(M_1, \circ_1, e_1)$  and  $(M_2, \circ_2, e_2)$ , with multiplication defined by*

$$(X_1, X_2) \circ (Y_1, Y_2) = (X_1 \circ_1 Y_1, X_2 \circ_2 Y_2), \quad (12)$$

*for any  $X_1, Y_1 \in \mathcal{X}(M_1)$  and  $X_2, Y_2 \in \mathcal{X}(M_2)$  (considered as vector fields on  $M_1 \times M_2$ ). If  $\mathcal{E}_1$  is an eventual identity on  $(M, \circ_1, e_1)$  and  $\mathcal{E}_2$  is an eventual identity on  $(M, \circ_2, e_2)$ , then  $\mathcal{E} := \mathcal{E}_1 + \mathcal{E}_2$  is an eventual identity on  $(M_1 \times M_2, \circ, e_1 + e_2)$ . Moreover, any eventual identity on  $(M_1 \times M_2, \circ, e_1 + e_2)$  is obtained this way.*

*Proof.* *i)* If  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are eventual identities then  $\mathcal{E}_1 \circ \mathcal{E}_2$  is invertible and for any  $X, Y \in \mathcal{X}(M)$ ,

$$\begin{aligned} L_{\mathcal{E}_1 \circ \mathcal{E}_2}(\circ)(X, Y) &= \mathcal{E}_1 \circ L_{\mathcal{E}_2}(\circ)(X, Y) + \mathcal{E}_2 \circ L_{\mathcal{E}_1}(\circ)(X, Y) \\ &= (\mathcal{E}_1 \circ [e, \mathcal{E}_2] + \mathcal{E}_2 \circ [e, \mathcal{E}_1]) \circ X \circ Y \\ &= [e, \mathcal{E}_1 \circ \mathcal{E}_2] \circ X \circ Y \end{aligned}$$

where in the last equality we used  $L_e(\circ) = 0$ . Moreover, from Lemma 4 and Theorem 3, if  $\mathcal{E}$  is an eventual identity then also  $\mathcal{E}^{-1}$  is an eventual identity. Our first claim follows.

*ii)* Recall the following relation proved in Proposition 4.3 of [8]: for any vector fields  $X, Y, Z, W \in \mathcal{X}(M)$ ,

$$\begin{aligned} L_{[X, Y]}(\circ)(Z, W) &= [X, L_Y(\circ)(Z, W)] - L_Y(\circ)([X, Z], W) - L_Y(\circ)(Z, [X, W]) \\ &\quad - [Y, L_X(\circ)(Z, W)] + L_X(\circ)([Y, Z], W) + L_X(\circ)(Z, [Y, W]). \end{aligned}$$

Our second claim follows this relation and Theorem 3.

*iii)* It is straightforward to check that a sum of eventual identities on the factors gives an eventual identity on the product  $(M_1 \times M_2, \circ, e_1 + e_2)$ . The converse is more involved and goes as follows (a similar argument has been used for the decomposition of Euler vector fields on product  $F$ -manifolds, see Theorem 2.11 of [7]). Let  $\mathcal{E}$  be an eventual identity on  $(M_1 \times M_2, \circ, e_1 + e_2)$  and define  $\mathcal{E}_k := e_k \circ \mathcal{E}$  for  $k \in \{1, 2\}$ . From (12)  $\mathcal{E}_k$  is tangent to  $M_k$  at any point of  $M_1 \times M_2$ . We will show that  $\mathcal{E}_1$  is a vector field on  $M_1$  (a similar argument shows that  $\mathcal{E}_2$  is a vector field on  $M_2$ ). For this, let  $Z$  be a vector field on  $M_2$ . Note that

$$L_{\mathcal{E}_1}(\circ)(Z, e_2) = \mathcal{E} \circ L_{e_1}(\circ)(Z, e_2) + e_1 \circ L_{\mathcal{E}}(\circ)(Z, e_2) = 0 \quad (13)$$

because  $L_{e_1}(\circ) = 0$  (easy check) and

$$e_1 \circ L_{\mathcal{E}}(\circ)(Z, e_2) = e_1 \circ [e, \mathcal{E}] \circ Z \circ e_2 = 0$$

where we used condition (3) on  $\mathcal{E}$  and  $e_1 \circ e_2 = 0$ . From (13) and  $Z = Z \circ e_2$  we get

$$[\mathcal{E}_1, Z] = L_{\mathcal{E}_1}(Z \circ e_2) = [\mathcal{E}_1, Z] \circ e_2 + Z \circ [\mathcal{E}_1, e_2].$$

It follows that  $[\mathcal{E}_1, Z]$  is tangent to  $M_2$  at any point of  $M_1 \times M_2$ . This holds for any vector field  $Z$  on  $M_2$  and hence  $\mathcal{E}_1$  is a vector field on  $M_1$ . Similarly,  $\mathcal{E}_2$  is a vector field on  $M_2$ . Since  $\mathcal{E}$  is invertible on  $(M, \circ, e_1 + e_2)$ ,  $\mathcal{E}_1$  is invertible on  $(M, \circ_1, e_1)$  and  $\mathcal{E}_2$  is invertible on  $(M, \circ_2, e_2)$ . From

$$[e, \mathcal{E}] = [e_1, \mathcal{E}_1] + [e_2, \mathcal{E}_2]$$

and

$$L_{\mathcal{E}}(\circ)(X, Y) = [e, \mathcal{E}] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M)$$

we get

$$L_{\mathcal{E}_k}(\circ_k)(X, Y) = [e_k, \mathcal{E}_k] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M_k), \quad k \in \{1, 2\},$$

i.e.  $\mathcal{E}_k$  is an eventual identity on the  $F$ -manifold  $(M_k, \circ_k, e_k)$ . Our claim follows.  $\square$

By a result of Hertling [7], any  $F$ -manifold locally decomposes into a product of irreducible  $F$ -manifolds. The decomposition of eventual identities on product  $F$ -manifolds into sums of eventual identities on the factors gives a compatibility between our duality for  $F$ -manifolds with eventual identities and Hertling's decomposition of  $F$ -manifolds, as follows.

**Theorem 6.** *Let  $(M, \circ, e)$  be an  $F$ -manifold with irreducible decomposition*

$$(M, \circ, e) \cong (M_1, \circ_1, e_1) \times \cdots \times (M_l, \circ_l, e_l) \quad (14)$$

*near a point  $p \in M$  and let  $\mathcal{E}$  be an eventual identity on  $(M, \circ, e)$ . Consider the decomposition*

$$\mathcal{E} = \mathcal{E}_1 + \cdots + \mathcal{E}_l \quad (15)$$

*of  $\mathcal{E}$  into a sum of eventual identities  $\mathcal{E}_k$  on the factors. Let  $(M, *, \mathcal{E}, e)$  be the dual of  $(M, \circ, e, \mathcal{E})$  and  $(M_k, *_k, \mathcal{E}_k, e_k)$  the dual of  $(M_k, \circ_k, e_k, \mathcal{E}_k)$ , for any  $1 \leq k \leq l$ . Then*

$$(M, *, \mathcal{E}) \cong (M_1, *_1, \mathcal{E}_1) \times \cdots \times (M_l, *_l, \mathcal{E}_l) \quad (16)$$

*is the irreducible decomposition of the  $F$ -manifold  $(M, *, \mathcal{E})$  near  $p$ .*

*Proof.* The decomposition (15) was proved in Proposition 5 iii). The decomposition (16) follows from (14) and (15).  $\square$

We end this Section with some more remarks and examples of eventual identities.

**Remark 7.** *i) Condition (3) which characterizes eventual identities is equivalent to the apparently weaker condition*

$$L_{\mathcal{E}}(\circ)(X, Y) = v \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M), \quad (17)$$

*for a vector field  $v$ . Indeed, if in relation (17) we replace  $X$  and  $Y$  by  $e$  we get  $v = L_{\mathcal{E}}(\circ)(e, e)$ . On the other hand,*

$$L_{\mathcal{E}}(\circ)(e, e) = [\mathcal{E}, e \circ e] - 2[\mathcal{E}, e] \circ e = [e, \mathcal{E}]$$

and hence  $v = [e, \mathcal{E}]$ , as in (3). In particular, any invertible Euler vector field  $E$  of weight  $d$  is an eventual identity and  $[e, E] = de$ .

ii) If  $\mathcal{E}$  is an eventual identity on an  $F$ -manifold  $(M, \circ, e)$ , then

$$[\mathcal{E}^n, \mathcal{E}^m] = (m - n)\mathcal{E}^{m+n-1} \circ [e, \mathcal{E}], \quad \forall m, n \in \mathbb{Z}. \quad (18)$$

The proof is by induction. When  $\mathcal{E}$  is Euler and  $m, n \geq 0$ , (18) was proved in [11] (see Theorem 5.6); when  $n = -1$  and  $m = 0$  (18) was proved in Lemma 4.

iii) Let  $(M, \circ, e)$  be a semi-simple  $F$ -manifold with canonical coordinates  $(u^1, \dots, u^n)$ , i.e.

$$\frac{\partial}{\partial u^i} \circ \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^j}, \quad \forall i, j$$

and

$$e = \frac{\partial}{\partial u^1} + \dots + \frac{\partial}{\partial u^n}.$$

Any eventual identity is of the form

$$\mathcal{E} = f_1 \frac{\partial}{\partial u^1} + \dots + f_n \frac{\partial}{\partial u^n},$$

where  $f_i$  are smooth non-vanishing functions depending only on  $u^i$ .

iv) Here is an example considered in [7], when the multiplication is not semi-simple. Let  $M := \mathbb{R}^2$  with multiplication defined by

$$\frac{\partial}{\partial x^1} \circ \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}, \quad \frac{\partial}{\partial x^2} \circ \frac{\partial}{\partial x^2} = 0, \quad i \in \{1, 2\}.$$

It can be checked that  $\circ$  defines an  $F$ -manifold structure and any eventual identity is of the form

$$\mathcal{E} = f_1 \frac{\partial}{\partial x^1} + f_2 \frac{\partial}{\partial x^2},$$

where  $f_1 = f_1(x^1)$  depends only on  $x^1$  and is non-vanishing.

### 3 Eventual identities and compatible metrics

The two metrics  $g$  and  $\tilde{g}$  on a Frobenius manifold have the important property that they form a flat pencil, that is, the metric  $g_\lambda^* := g^* + \lambda \tilde{g}^*$  is flat, for all values of  $\lambda$ . This condition results, via the Dubrovin-Novikov theorem, to a bi-Hamiltonian structure. What is important in this construction is not the flatness of the metrics but their compatibility. Curved metrics can, via Ferapontov's extension of the Dubrovin-Novikov theorem, define (non-local) Hamiltonian structures but it is the compatibility of two such metrics that will ensure a (non-local) bi-Hamiltonian structure. In this Section we construct compatible pair of metrics on  $F$ -manifolds with eventual identities.

We begin by recalling basic definitions and results on compatible pair of metrics. First we fix the conventions we will use in this and the following Sections.



**Conventions 8.** Let  $g$  and  $\tilde{g}$  be two metrics on a manifold  $M$ , with associated pencil of inverse metrics  $g_\lambda^* := g^* + \lambda \tilde{g}^*$  (assumed to be non-degenerate for any  $\lambda$ ). We denote by  $g : TM \rightarrow T^*M$ ,  $X \rightarrow g(X)$  and  $g^* : T^*M \rightarrow TM$ ,  $\alpha \rightarrow g^*(\alpha)$  the isomorphisms defined by raising and lowering indices using  $g$  and similar notations will be used for the isomorphisms between  $TM$  and  $T^*M$  defined by  $\tilde{g}$  and  $g_\lambda$ . To simplify notations we shall often denote by  $X^\flat = \tilde{g}(X)$  the dual 1-form of a vector field  $X$  with respect to  $\tilde{g}$  (it is important to note that  $X^\flat$  is the dual 1-form using  $\tilde{g}$  and not  $g$ , since the metrics  $g$  and  $\tilde{g}$  will not play symmetric roles). The Levi-Civita connections of  $g$ ,  $g_\lambda$  and  $\tilde{g}$  will be denoted by  $\nabla$ ,  $\nabla^\lambda$  and  $\tilde{\nabla}$  respectively;  $R^g$ ,  $R^\lambda$  and  $R^{\tilde{g}}$  and will denote the curvatures of  $g$ ,  $g_\lambda$  and  $\tilde{g}$ .

**Definition 9.** *i) A pair  $(g, \tilde{g})$  is called almost compatible if*

$$g_\lambda^*(\nabla_X^\lambda \alpha) = g^*(\nabla_X \alpha) + \lambda \tilde{g}^*(\tilde{\nabla}_X \alpha)$$

*for any  $X \in \mathcal{X}(M)$ ,  $\alpha \in \Omega^1(M)$  and  $\lambda$  constant.*

*ii) A pair  $(g, \tilde{g})$  is called compatible if  $(g, \tilde{g})$  are almost compatible and*

$$g_\lambda^*(R_{X,Y}^\lambda \alpha) = g^*(R_{X,Y}^g \alpha) + \lambda \tilde{g}^*(R_{X,Y}^{\tilde{g}} \alpha) \quad (19)$$

*for any  $X, Y \in \mathcal{X}(M)$ ,  $\alpha \in \Omega^1(M)$  and  $\lambda$  constant.*

According to [13] (see also [2] for a shorter proof) the metrics  $(g, \tilde{g})$  are almost compatible if and only if the Nijenhuis tensor of  $A := g^* \tilde{g} \in \text{End}(TM)$ , defined by

$$N_A(X, Y) = -[AX, AY] + A([AX, Y] + [X, AY]) - A^2[X, Y], \quad X, Y \in \mathcal{X}(M)$$

is identically zero. Moreover, according to Theorem 3.1 of [2], if  $(g, \tilde{g})$  are almost compatible then  $(g, \tilde{g})$  are compatible if and only if one of the following equivalent conditions holds:

$$g^*(\tilde{\nabla}_Y \alpha - \nabla_Y \alpha, \tilde{\nabla}_X \beta - \nabla_X \beta) = g^*(\tilde{\nabla}_X \alpha - \nabla_X \alpha, \tilde{\nabla}_Y \beta - \nabla_Y \beta) \quad (20)$$

or

$$\tilde{g}^*(\tilde{\nabla}_Y \alpha - \nabla_Y \alpha, \tilde{\nabla}_X \beta - \nabla_X \beta) = \tilde{g}^*(\tilde{\nabla}_X \alpha - \nabla_X \alpha, \tilde{\nabla}_Y \beta - \nabla_Y \beta), \quad (21)$$

for any vector fields  $X, Y \in \mathcal{X}(M)$  and 1-forms  $\alpha, \beta \in \Omega^1(M)$ .

We now turn to  $F$ -manifolds and we show in Proposition 10 bellow that an eventual identity on an  $F$ -manifold together with a (multiplication) invariant metric determine a pair of almost compatible metrics. A metric  $\tilde{g}$  on an  $F$ -manifold  $(M, \circ, e)$  is called invariant if

$$\tilde{g}(X \circ Y, Z) = \tilde{g}(X, Y \circ Z), \quad \forall X, Y, Z \in \mathcal{X}(M)$$

or

$$\tilde{g}(X, Y) = \epsilon(X \circ Y).$$

where  $\epsilon = \tilde{g}(e)$  is the coidentity. Thus  $\tilde{g}$  is uniquely determined by the coidentity  $\epsilon \in \Omega^1(M)$  and invariant metrics on  $(M, \circ, e)$  are in bijective correspondence with 1-forms on  $M$ .

**Proposition 10.** *Let  $(M, \circ, e, \tilde{g}, \mathcal{E})$  be an  $F$ -manifold together with an invariant metric  $\tilde{g}$  and eventual identity  $\mathcal{E}$ . Define a new metric  $g$  by*

$$g(X, Y) = \tilde{g}(\mathcal{E}^{-1} \circ X, Y), \quad \forall X, Y \in \mathcal{X}(M). \quad (22)$$

*Then  $(g, \tilde{g})$  are almost compatible.*

*Proof.* From (22),

$$g^* \tilde{g}(X) = \mathcal{E} \circ X, \quad \forall X \in TM$$

Using the  $F$ -manifold condition (1) together with the characterization (3) of eventual identities, we get:

$$\begin{aligned} N_{\mathcal{E} \circ}(X, Y) &= -L_{\mathcal{E} \circ X}(\mathcal{E} \circ Y) + \mathcal{E} \circ (L_X(\mathcal{E} \circ Y) - L_Y(\mathcal{E} \circ X)) - \mathcal{E}^2 \circ [X, Y] \\ &= -[\mathcal{E} \circ X, \mathcal{E}] \circ Y - [\mathcal{E} \circ X, Y] \circ \mathcal{E} - L_{\mathcal{E} \circ X}(\circ)(\mathcal{E}, Y) \\ &\quad + \mathcal{E} \circ ([X, \mathcal{E}] \circ Y + \mathcal{E} \circ [X, Y] + L_X(\circ)(\mathcal{E}, Y) - [Y, \mathcal{E}] \circ X) \\ &\quad - \mathcal{E}^2 \circ [Y, X] - \mathcal{E} \circ L_Y(\circ)(\mathcal{E}, X) - \mathcal{E}^2 \circ [X, Y] \\ &= L_{\mathcal{E}}(\mathcal{E} \circ X) \circ Y + L_Y(\mathcal{E} \circ X) \circ \mathcal{E} - \mathcal{E} \circ L_X(\circ)(\mathcal{E}, Y) \\ &\quad - X \circ L_{\mathcal{E}}(\circ)(\mathcal{E}, Y) + \mathcal{E} \circ Y \circ [X, \mathcal{E}] + \mathcal{E}^2 \circ [X, Y] \\ &\quad + \mathcal{E} \circ L_X(\circ)(\mathcal{E}, Y) - \mathcal{E} \circ X \circ [Y, \mathcal{E}] - \mathcal{E}^2 \circ [Y, X] \\ &\quad - \mathcal{E} \circ L_Y(\circ)(\mathcal{E}, X) - \mathcal{E}^2 \circ [X, Y] \\ &= L_{\mathcal{E}}(\circ)(\mathcal{E}, X) \circ Y - L_{\mathcal{E}}(\circ)(\mathcal{E}, Y) \circ X \\ &= [e, \mathcal{E}] \circ \mathcal{E} \circ (X \circ Y - Y \circ X) = 0, \end{aligned}$$

for any vector fields  $X, Y \in \mathcal{X}(M)$ . Our claim follows.  $\square$

When the  $F$ -manifold  $(M, \circ, e)$  is semi-simple, the pair  $(g, \tilde{g})$  of Proposition 10 is semi-simple as well and, being almost compatible,  $(g, \tilde{g})$  is automatically compatible [13, 2]. Without the semi-simplicity assumption, the pair  $(g, \tilde{g})$  is not always compatible. We are going to show that  $(g, \tilde{g})$  is compatible (without the semi-simplicity assumption), provided that the coidentity associated to  $\tilde{g}$  is closed. To simplify terminology we introduce the following definition.

**Definition 11.** *An almost Riemannian  $F$ -manifold is an  $F$ -manifold  $(M, \circ, e, \tilde{g})$  together with an invariant metric  $\tilde{g}$  such that the coidentity  $\epsilon \in \Omega^1(M)$  defined by*

$$\epsilon(X) := \tilde{g}(e, X), \quad \forall X \in TM$$

*is closed.*

There is a result of Hertling [7], which states that the closeness of the coidentity  $\epsilon$  on an  $F$ -manifold  $(M, \circ, e, \tilde{g})$  with invariant metric  $\tilde{g}$  is equivalent with the total symmetry of the  $(4, 0)$ -tensor field

$$(\tilde{\nabla} \circ)(X, Z, Y, V) := \tilde{g}(\tilde{\nabla}_X(\circ)(Z, Y), V), \quad (23)$$

or to the symmetry in the first two arguments (the symmetry in the last three arguments being a consequence of the invariance of  $\tilde{g}$ ).

**Theorem 12.** *Let  $(M, \circ, e, \tilde{g}, \mathcal{E})$  be an almost Riemannian  $F$ -manifold with eventual identity  $\mathcal{E}$ . Define a new metric  $g$  by*

$$g(X, Y) = \tilde{g}(\mathcal{E}^{-1} \circ X, Y), \quad \forall X, Y \in \mathcal{X}(M).$$

*Then  $(g, \tilde{g})$  are compatible.*

*Proof.* From Proposition 10, the metrics  $(g, \tilde{g})$  are almost compatible. To prove that  $(g, \tilde{g})$  are compatible, it is enough to show that (21) is satisfied (see our comments above). The Koszul formula for the Levi-Civita  $\nabla$  of  $g$  translated to  $T^*M$  gives

$$2g^*(\nabla_Y \alpha, \beta) = -g^*(i_Y d\beta, \alpha) + g^*(i_Y d\alpha, \beta) + Yg^*(\alpha, \beta) - g([g^* \alpha, g^* \beta], Y), \quad (24)$$

where  $\alpha, \beta \in \Omega^1(M)$  and  $Y \in \mathcal{X}(M)$ . A similar expression holds for the Levi-Civita connection  $\tilde{\nabla}$  of  $\tilde{g}$  on  $T^*M$ :

$$2\tilde{g}^*(\tilde{\nabla}_Y \alpha, \beta) = -\tilde{g}^*(i_Y d\beta, \alpha) + \tilde{g}^*(i_Y d\alpha, \beta) + Y\tilde{g}^*(\alpha, \beta) - \tilde{g}([\tilde{g}^* \alpha, \tilde{g}^* \beta], Y). \quad (25)$$

Combining (24) and (25) and using that  $(g, \tilde{g})$  are almost compatible we get, by the argument of Proposition 5.10 of [2],

$$2g^*(\nabla_Y X^b - \tilde{\nabla}_Y X^b, Z^b) = (L_{\mathcal{E}} \tilde{g})(X \circ Y, Z) + \tilde{g}([e, \mathcal{E}] \circ X - 2\tilde{\nabla}_X \mathcal{E} \circ Y, Z) \quad (26)$$

where  $X^b, Z^b \in \Omega^1(M)$  correspond to  $X, Z \in \mathcal{X}(M)$  using the duality defined by  $\tilde{g}$ . Now, for a vector field  $V$ , define a 1-form  $(L_{\mathcal{E}} \tilde{g})(V)$  by

$$(L_{\mathcal{E}} \tilde{g})(V)(Z) := (L_{\mathcal{E}} \tilde{g})(V, Z), \quad \forall Z \in \mathcal{X}(M).$$

With this notation,

$$(L_{\mathcal{E}} \tilde{g})(X \circ Y, Z) = (L_{\mathcal{E}} \tilde{g})(X \circ Y)(Z).$$

Since  $L_{\mathcal{E}} \tilde{g}$  is multiplication invariant (this follows by taking the Lie derivative with respect to  $\mathcal{E}$  of  $\tilde{g}(X \circ Y, Z) = \tilde{g}(X, Y \circ Z)$  and using condition (3) on  $\mathcal{E}$ ), we obtain

$$(L_{\mathcal{E}} \tilde{g})(X \circ Y) = X^b \circ Y^b \circ (L_{\mathcal{E}} \tilde{g})(e) \quad (27)$$

where  $\circ$  is the induced multiplication on  $T^*M$ , obtained by identifying  $TM$  with  $T^*M$  using  $\tilde{g}$ . Denoting  $\alpha := X^b$ , from (26) and (27) we get

$$2(\nabla_Y \alpha - \tilde{\nabla}_Y \alpha) = Y^b \circ \mathcal{E}^{-1, b} \circ \left( ((L_{\mathcal{E}} \tilde{g})(e) + [e, \mathcal{E}]^b) \circ \alpha - 2\tilde{\nabla}_{\tilde{g}^* \alpha} \mathcal{E}^b \right). \quad (28)$$

Since  $\tilde{g}$  is invariant,  $\tilde{g}^*$  is also invariant (with respect to  $\circ$  on  $T^*M$ ) and relation (28) implies that (21) is satisfied. Being almost compatible, the metrics  $(g, \tilde{g})$  are compatible.  $\square$

We end this Section by making some comments on Theorem 12. Similar results were proved in [2], with the almost Riemannian  $F$ -manifold replaced by a weak  $\mathcal{F}$ -manifold  $(M, \circ, e, \tilde{g}, E)$ , i.e. the multiplication  $\circ$  on  $TM$  is commutative associative with unity field  $e$ ,  $\tilde{g}$  is an invariant metric,  $E$  is an invertible Euler vector field which is also conformal-Killing with respect to  $\tilde{g}$  and the weak symmetry condition

$$(\tilde{\nabla} \circ)(E, Z, Y, V) = (\tilde{\nabla} \circ)(Z, E, Y, V), \quad \forall Y, Z, V \in \mathcal{X}(M) \quad (29)$$

holds; in general,  $\circ$  does not satisfy the integrability condition (1), so a weak  $\mathcal{F}$ -manifold is not always an  $F$ -manifold. We are going to show that a weak  $\mathcal{F}$ -manifold which is also an  $F$ -manifold is an almost Riemannian  $F$ -manifold. Thus, in the setting of  $F$ -manifolds, Theorem 12 extends the statement about the compatibility of metrics in Theorem 5.8 of [2], by replacing the Euler vector field with an eventual identity.

**Lemma 13.** *Let  $(M, \circ, e, \mathcal{E}, \tilde{g})$  be an  $F$ -manifold together with an invertible vector field  $\mathcal{E}$  and invariant metric  $\tilde{g}$ . Assume the weak symmetry condition*

$$(\tilde{\nabla} \circ)(\mathcal{E}, Z, Y, V) = (\tilde{\nabla} \circ)(Z, \mathcal{E}, Y, V), \quad \forall Y, Z, V \in \mathcal{X}(M) \quad (30)$$

*holds. Then  $(M, \circ, e, \tilde{g})$  is an almost Riemannian  $F$ -manifold.*

*Proof.* We need to show that the coidentity  $\epsilon = \tilde{g}(e)$  is closed. It is known that on any  $F$ -manifold  $(M, \circ, e, \tilde{g})$  with multiplication  $\circ$ , unity field  $e$ , invariant metric  $\tilde{g}$  and coidentity  $\epsilon$ , the tensor fields  $\tilde{\nabla} \circ$  and  $d\epsilon$  are related by the following identity (see the proof of Theorem 2.15 of [7]):

$$2(\tilde{\nabla} \circ)(X, Z, Y, V) - 2(\tilde{\nabla} \circ)(Z, X, Y, V) = d\epsilon(Y \circ Z, X \circ V) - d\epsilon(X \circ Y, Z \circ V). \quad (31)$$

Taking  $X := \mathcal{E}$  in (31) and using our hypothesis we get

$$d\epsilon(\mathcal{E} \circ Y, Z \circ V) = d\epsilon(Y \circ Z, \mathcal{E} \circ V). \quad (32)$$

With  $Z := e$ , (32) becomes

$$d\epsilon(\mathcal{E} \circ Y, V) = d\epsilon(Y, \mathcal{E} \circ V). \quad (33)$$

Replacing in (33)  $V$  by  $V \circ Z$  and using again (32) we get

$$d\epsilon(Y, \mathcal{E} \circ V \circ Z) = d\epsilon(\mathcal{E} \circ Y, V \circ Z) = d\epsilon(Y \circ Z, \mathcal{E} \circ V). \quad (34)$$

Since  $\mathcal{E}$  is invertible, relation (34) is equivalent to

$$d\epsilon(Y, Z \circ V) = d\epsilon(Y \circ Z, V), \quad \forall Y, Z, V \in \mathcal{X}(M), \quad (35)$$

i.e.  $d\epsilon$  is multiplication invariant. Being skew-symmetric,  $d\epsilon = 0$ . Our claim follows.  $\square$

## 4 Duality and Riemannian $F$ -manifolds

Riemannian  $F$ -manifolds were first introduced in the literature in [10]. In this Section we prove that the class of Riemannian  $F$ -manifolds is preserved by the duality between  $F$ -manifolds with eventual identities. In the next Section we apply this result to the theory of integrable systems.

**Definition 14.** *A Riemannian  $F$ -manifold is an  $F$ -manifold  $(M, \circ, e, \tilde{g})$  together with an invariant metric  $\tilde{g}$  such that:*

*i) the coidentity  $\epsilon = \tilde{g}(e) \in \Omega^1(M)$  is closed, i.e.  $(M, \circ, e, \tilde{g})$  is an almost Riemannian  $F$ -manifold.*

ii) the curvature condition

$$Z \circ R^{\tilde{g}}(V, Y)(X) + Y \circ R^{\tilde{g}}(Z, V)(X) + V \circ R^{\tilde{g}}(Y, Z)(X) = 0, \quad (36)$$

is satisfied, for any  $X, Y, Z, V \in \mathcal{X}(M)$ .

Our main result in this Section is the following Theorem.

**Theorem 15.** *Let  $(M, \circ, e, \tilde{g}, \mathcal{E})$  be an  $F$ -manifold with invariant metric  $\tilde{g}$  and eventual identity  $\mathcal{E}$ . Define a second metric  $g$  by*

$$g(X, Y) = \tilde{g}(\mathcal{E}^{-1} \circ X, Y), \quad \forall X, Y \in \mathcal{X}(M) \quad (37)$$

and let  $(M, *, \mathcal{E}, e)$  be the dual of  $(M, \circ, e, \tilde{g})$ . Then  $(M, \circ, e, \tilde{g})$  is a Riemannian  $F$ -manifold if and only if  $(M, *, \mathcal{E}, g)$  is a Riemannian  $F$ -manifold.

*Proof.* From (37), the coidentities of  $(M, \circ, e, \tilde{g})$  and  $(M, *, \mathcal{E}, g)$  coincide. Thus  $(M, \circ, e, \tilde{g})$  is an almost Riemannian  $F$ -manifold if and only if  $(M, *, \mathcal{E}, g)$  is an almost Riemannian  $F$ -manifold.

Assume now that  $(M, \circ, e, \tilde{g})$  is a Riemannian  $F$ -manifold. By our comments from the previous Section, the tensor field  $\tilde{\nabla} \circ$  is totally symmetric. With the conventions from the proof of Theorem 12, the total symmetry of  $\tilde{\nabla} \circ$  and relation (28), together with an easy curvature computation show that the curvatures of  $g$  and  $\tilde{g}$  on  $T^*M$  are related by

$$R^g(X, Y)(\alpha) = R^{\tilde{g}}(X, Y)(\alpha) + Q(\alpha, Y) \circ X^{\flat} - Q(\alpha, X) \circ Y^{\flat}, \quad (38)$$

where

$$Q(\alpha, X) := \mathcal{S}(\mathcal{S}(\alpha) \circ X^{\flat}) - \tilde{\nabla}_X(\mathcal{S})(\alpha), \quad \forall \alpha \in T^*M, \quad \forall X \in TM$$

and

$$\mathcal{S}(\alpha) := \frac{1}{2} \mathcal{E}^{-1, \flat} \circ \left( ((L_{\mathcal{E}} \tilde{g})(e) + [e, \mathcal{E}]^{\flat}) \circ \alpha - 2 \tilde{\nabla}_{\tilde{g}^* \alpha} \mathcal{E}^{\flat} \right). \quad (39)$$

(Recall that  $TM$  and  $T^*M$  are identified using  $\tilde{g}$  and  $\circ$  above denotes the induced multiplication on  $T^*M$ ). Since  $(M, \circ, e, \tilde{g})$  is a Riemannian  $F$ -manifold, relation (36) holds. Translated to  $T^*M$ , it gives

$$Z^{\flat} \circ R^{\tilde{g}}(V, Y)(\alpha) + Y^{\flat} \circ R^{\tilde{g}}(Z, V)(\alpha) + V^{\flat} \circ R^{\tilde{g}}(Y, Z)(\alpha) = 0, \quad (40)$$

for any vector fields  $Y, Z$  and  $V$  and covector  $\alpha$ . Using (38), relation (40) becomes

$$Z^{\flat} \circ R^g(V, Y)(\alpha) + Y^{\flat} \circ R^g(Z, V)(\alpha) + V^{\flat} \circ R^g(Y, Z)(\alpha) = 0. \quad (41)$$

Take in (41)  $\alpha := g(X)$ . Note that

$$Z^{\flat} \circ R^g(V, Y)(\alpha) = Z^{\flat} \circ g(R^g(V, Y)(X)) = Z^{\flat} \circ \mathcal{E}^{-1, \flat} \circ R^g(V, Y)(X)^{\flat}$$

and similarly for  $Y^{\flat} \circ R^g(Z, V)(\alpha)$  and  $V^{\flat} \circ R^g(Y, Z)(\alpha)$ . On  $TM$  relation (41) becomes

$$\mathcal{E}^{-1} \circ (Z \circ R^g(V, Y)(X) + Y \circ R^g(Z, V)(X) + V \circ R^g(Y, Z)(X)) = 0 \quad (42)$$

for any vector fields  $X, Y, Z, V$ , or

$$Z * R^g(V, Y)(X) + Y * R^g(Z, V)(X) + V * R^g(Y, Z)(X) = 0. \quad (43)$$

We proved that  $(M, *, \mathcal{E}, g)$  is a Riemannian  $F$ -manifold. Our claim follows.  $\square$

## 5 Applications to integrable systems

There is a close relationship between  $F$ -manifolds and the theory of integrable systems of hydrodynamic type. In particular we draw together various results of [10] into the following theorem.

**Theorem 16.** *Consider an almost Riemannian  $F$ -manifold  $(M, \circ, e, \tilde{g})$ . If  $\tilde{X}$  and  $\tilde{Y}$  are two vector fields which satisfy the condition*

$$(\tilde{\nabla}_Z \tilde{X}) \circ V = (\tilde{\nabla}_V \tilde{X}) \circ Z \quad \forall V, Z \in \mathcal{X}(M) \quad (44)$$

*then the associated flows*

$$\begin{aligned} U_t &= \tilde{X} \circ U_x, \\ U_\tau &= \tilde{Y} \circ U_x \end{aligned}$$

*commute. Moreover, for arbitrary vector fields  $Y, V, Z \in \mathcal{X}(M)$  the identity*

$$Z \circ R^{\tilde{g}}(V, Y)(\tilde{X}) + V \circ R^{\tilde{g}}(Y, Z)(\tilde{X}) + Y \circ R^{\tilde{g}}(Z, V)(\tilde{X}) = 0$$

*holds for any solution  $\tilde{X}$  of (44).*

By twisting solutions  $\tilde{X}$  of (44) by an eventual identity one may derive the dual, or twisted, version of the above theorem.

**Lemma 17.** *Let  $(M, \circ, e, \tilde{g})$  be an almost Riemannian  $F$ -manifold and  $\tilde{X} \in \mathcal{X}(M)$  a vector field such that*

$$\tilde{\nabla}_Y \tilde{X} \circ V = \tilde{\nabla}_V \tilde{X} \circ Y, \quad \forall Y, V \in \mathcal{X}(M). \quad (45)$$

*Let  $\mathcal{E}$  be an eventual identity on  $(M, \circ, e)$  and  $(M, *, \mathcal{E}, g)$  the dual almost Riemannian  $F$ -manifold, like in Theorem 15. Then  $X = \tilde{X} \circ \mathcal{E}$  satisfies the dual equation*

$$(\nabla_Y X) * V = (\nabla_V X) * Y, \quad \forall Y, V \in \mathcal{X}(M). \quad (46)$$

*Proof.* Recall, from relation (28), that

$$\nabla_Y \alpha = \tilde{\nabla}_Y \alpha + \mathcal{S}(\alpha) \circ Y^b, \quad \forall Y \in TM \quad (47)$$

where  $\mathcal{S}(\alpha)$  is given by (39), as usual  $Y^b = \tilde{g}(Y)$  and  $\circ$  is the induced multiplication on  $T^*M$ , obtained by identifying  $TM$  with  $T^*M$  using  $\tilde{g}$ . In (47) let  $\alpha := \tilde{X}^b = g(\tilde{X} \circ \mathcal{E})$ . Relation (47) becomes

$$g(\nabla_Y (\tilde{X} \circ \mathcal{E})) = \tilde{\nabla}_Y \tilde{X}^b + Y^b \circ \mathcal{S}(\tilde{X}^b) \quad (48)$$

Applying  $\tilde{g}^*$  to (48) and using  $(\tilde{g}^* g)(X) = \mathcal{E}^{-1} \circ X$  we get

$$\nabla_Y (\tilde{X} \circ \mathcal{E}) = \mathcal{E} \circ \tilde{\nabla}_Y \tilde{X} + \mathcal{E} \circ Y \circ \tilde{g}^*(\mathcal{S}(\tilde{X}^b)). \quad (49)$$

From (49) we get

$$\nabla_Y (\tilde{X} \circ \mathcal{E}) * V = \tilde{\nabla}_Y \tilde{X} \circ V + Y \circ V \circ \tilde{g}^*(\mathcal{S}(\tilde{X}^b)),$$

which, from (45), is symmetric in  $Y$  and  $V$ . Relation (46) is satisfied.  $\square$

Thus we obtain dual flow equations

$$\begin{aligned} U_t &= X * U_x, \\ U_\tau &= Y * U_x \end{aligned}$$

from vector fields  $\tilde{X}, \tilde{Y} \in \mathcal{X}(M)$  satisfying (44) by twisting by an eventual identity. Moreover by Theorem 15 the dual curvature condition also holds.

This duality, or twisting, by an eventual identity gives a geometric form of certain well-known arguments from the theory of integrable systems of hydrodynamic type which originate in the work of Tsarev. Recall that in the semi-simple case the basic equation  $U_t = \tilde{X} \circ U_x$  reduces to diagonal form

$$u_t^i = \tilde{X}^i(\mathbf{u}) u_x^i$$

so the components of  $\tilde{X}$  become the characteristic velocities of the quasilinear system. Equation (44) reduces to Tsarev's equation

$$\frac{\partial}{\partial u^i} \log \sqrt{g_{jj}} = \frac{\partial_i \tilde{X}^j}{\tilde{X}^i - \tilde{X}^j}, \quad i \neq j. \quad (50)$$

The integrability conditions for this system form the so-called semi-Hamiltonian conditions, which in turn are the coordinate form of (36).

Solutions of (50) possess a functional freedom: if  $\tilde{g}_{ii}(\mathbf{u})$  is a solution so is  $\tilde{g}_{ii}(\mathbf{u})/f_i(u^i)$ . This functional freedom can now be reinterpreted, via Remark 7 *iii*) on the form of eventual identities in the semi-simple case, as the dual version of the theory. Also since the  $f_i$  are arbitrary, one may replace it by  $f_i \rightarrow f_i + \lambda$  for any constant  $\lambda$ . Thus one recovers the pencil property  $g_\lambda^* = g^* + \lambda \tilde{g}^*$  and hence, by Proposition 10, a compatible pair of metrics and (non-local) bi-Hamiltonian structures (this last stage, from almost compatible to compatible being automatic in the semi-simple case).

In applications, where one is interested in finding bi-Hamiltonian structures for a specific system of equations, one tries to find a suitable eventual identity so that the metric  $g$  has simple curvature properties, such as flatness or constant curvature. If flat one arrives, via the original Dubrovin-Novikov theorem, at a local Hamiltonian structure. The simplest case is where both metrics are flat, and hence form a flat pencil and a local bi-Hamiltonian structure. With extra conditions one can arrive at a Frobenius manifold [5].

## 6 Duality and $tt^*$ -geometry

An holomorphic  $F$ -manifold is a complex manifold  $M$  together with an associative, commutative, with unity multiplication  $\circ$  on the sheaf of holomorphic vector fields, satisfying the  $F$ -manifold condition (1). Euler vector fields, identities, eventual identities etc are holomorphic and are defined like in the smooth case. In particular, our characterization of eventual identities developed in Theorem 3 holds also in the holomorphic setting.

In the same framework like in Sections 3 and 4, we add structures - hermitian metrics and real structures - on an holomorphic  $F$ -manifold  $(M, \circ, e)$  and we study their behaviour under twisting with an eventual identity. We assume that these structures are compatible with the multiplication  $\circ$ , i.e. in the terminology of [15] they form harmonic Higgs bundles or  $DChk$ -structures and

we determine necessary and sufficient conditions on the eventual identity such that the resulting dual structures are compatible in the same way. Harmonic Higgs bundles and  $DChk$ -structures are part of the so called CV-structures, introduced for the first time by Cecotti and Vafa in [1] and further studied in the literature, see [6], [14].

First we fix our conventions in the holomorphic setting.

**Conventions 18.** In this Section  $M$  will denote a complex manifold, considered as a smooth manifold together with an integrable complex structure  $J$ . Its real tangent bundle will be denoted  $TM$ . The sheaf of smooth real vector fields on  $(M, J)$  will be denoted as always by  $\mathcal{X}(M)$ , the sheaf of vector fields of type  $(1, 0)$  by  $\mathcal{T}_M^{1,0}$  and the sheaf of holomorphic vector fields by  $\mathcal{T}_M$ . A multiplication on the holomorphic tangent bundle  $T^{1,0}M$  will be trivially extended to the complexified bundle  $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ .

Following [4], [15], [7] we give the following definition, which recalls basic notions from the theory of  $tt^*$ -geometry.

**Definition 19.** *i) A pair  $(\tilde{g}, \tilde{h})$  formed by a complex bilinear, non-degenerate symmetric form  $\tilde{g}$  and a hermitian metric  $\tilde{h}$  on  $T^{1,0}M$  is called compatible if the Chern connection  $\tilde{D}$  of the holomorphic hermitian vector bundle  $(T^{1,0}M, \tilde{h})$  preserves  $\tilde{g}$ , i.e.  $\tilde{D}\tilde{g} = 0$ .*

*ii) Let  $\tilde{h}$  be a hermitian metric and  $\circ$  a commutative, associative, multiplication with unity field  $e$ , on  $T^{1,0}M$ . Define a Higgs field  $\tilde{C} \in \Omega^{1,0}(M, \text{End}(T^{1,0}M))$  by*

$$\tilde{C}_X Y := X \circ Y.$$

*The hermitian metric  $\tilde{h}$  on the Higgs bundle  $(T^{1,0}M, \tilde{C})$  is called harmonic (and  $(T^{1,0}M, \tilde{C}, \tilde{h})$  is a harmonic Higgs bundle) if  $\tilde{C}_X Y \in \mathcal{T}_M$ , for any  $X, Y \in \mathcal{T}_M$  and the  $tt^*$ -equations*

$$(\partial^{\tilde{D}} \tilde{C})_{X,Y} := \tilde{D}_X(\tilde{C}_Y) - \tilde{D}_Y(\tilde{C}_X) - \tilde{C}_{[X,Y]} = 0 \quad (51)$$

*and*

$$R_{X,\bar{Y}}^{\tilde{D}} + [\tilde{C}_X, \tilde{C}_{\bar{Y}}^{\flat}] = 0 \quad (52)$$

*are satisfied, for any  $X, Y \in \mathcal{T}_M^{1,0}$ . Above  $R^{\tilde{D}}$  denotes the curvature of the Chern connection  $\tilde{D}$  of  $(T^{1,0}M, \tilde{h})$  and  $\tilde{C}^{\flat}$  is the adjoint of  $\tilde{C}$  with respect to  $\tilde{h}$ , i.e.*

$$\tilde{h}(\tilde{C}_X Y, Z) = \tilde{h}(Y, \tilde{C}_X^{\flat} Z), \quad \forall Y, Z \in T^{1,0}M, \quad \forall X \in T_{\mathbb{C}}M.$$

*iii) Let  $(T^{1,0}M, \tilde{C}, \tilde{h})$  be a harmonic Higgs bundle and  $\tilde{k}$  a real structure on  $T^{1,0}M$  such that the complex bilinear form*

$$\tilde{g}(X, Y) := \tilde{h}(X, \tilde{k}Y)$$

*on  $T^{1,0}M$  is symmetric and (multiplication) invariant. The data  $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k})$  is called a  $DChk$ -structure if the pair  $(\tilde{g}, \tilde{h})$  is compatible.*

We remark that a harmonic Higgs bundle  $(T^{1,0}M, \tilde{C}, \tilde{h})$  has an associated pencil of flat connections

$$\tilde{D}^z := \tilde{D} + \frac{1}{z} \tilde{C} + z \tilde{C}^{\flat}. \quad (53)$$



The flatness property of this pencil encodes the entire geometry of the harmonic Higgs bundle [7].

For the remaining part of this Section we fix an  $F$ -manifold  $(M, \circ, e)$  together with an eventual identity  $\mathcal{E}$ , hermitian metric  $\tilde{h}$ , and real structure  $\tilde{k}$  on  $T^{1,0}M$  such that the complex bilinear form

$$\tilde{g}(X, Y) := \tilde{h}(X, \tilde{k}Y)$$

on  $T^{1,0}M$  is symmetric and invariant. Let

$$X * Y := X \circ Y \circ \mathcal{E}^{-1} \quad (54)$$

be the dual multiplication, with associated Higgs field

$$C_X Y := X \circ Y \circ \mathcal{E}^{-1}.$$

Assume that the inverse  $\mathcal{E}^{-1}$  has a square root  $\mathcal{E}^{-1/2}$  and define a new hermitian metric

$$h(X, Y) := \tilde{h}(\mathcal{E}^{-1/2} \circ X, \mathcal{E}^{-1/2} \circ Y) \quad (55)$$

and a new real structure

$$k(X) := \mathcal{E}^{1/2} \circ \tilde{k}(\mathcal{E}^{-1/2} \circ X)$$

on  $T^{1,0}M$ . It is straightforward to check that

$$g(X, Y) := h(X, kY) = \tilde{g}(\mathcal{E}^{-1/2} \circ X, \mathcal{E}^{-1/2} \circ Y). \quad (56)$$

In particular,  $g$  is symmetric, complex bilinear and invariant.

While in the smooth case it was not immediately clear that compatibility is preserved under twisting with eventual identities, the analogous statement in the holomorphic setting comes for free (and in fact holds under the weaker assumption that  $\mathcal{E}$  is holomorphic and invertible, not necessarily an eventual identity).

**Lemma 20.** *If the pair  $(\tilde{g}, \tilde{h})$  is compatible, then also the pair  $(g, h)$  is compatible.*

*Proof.* From (55), the Chern connections  $D$  and  $\tilde{D}$  of  $(T^{1,0}M, h)$  and  $(T^{1,0}M, \tilde{h})$  respectively are related by

$$D_X Z := \mathcal{E}^{1/2} \circ \tilde{D}_X(\mathcal{E}^{-1/2} \circ Z), \quad \forall X \in \mathcal{X}(M), \quad Z \in \mathcal{T}_M^{1,0}. \quad (57)$$

From (56) and (57),  $\tilde{D}\tilde{g} = 0$  if and only if  $Dg = 0$ .  $\square$

Note that if  $M$  is a Frobenius manifold with Euler vector field  $E$  then the choice  $\mathcal{E} = E$  results in a compatible pair  $(g, h)$  with certain special properties. The metric  $g$  is the intersection form of the manifold, and hence is flat. Thus there exists a distinguished coordinate system of so-called flat coordinates in which the components of  $g$  are constant. The metric  $h$  is then a natural hermitian metric defined on the complement of the classical discriminant  $\Sigma$  of the manifold.

**Theorem 21.** *i) Assume that  $\partial^{\tilde{D}}\tilde{C} = 0$ . Then  $\partial^D C = 0$  if and only if for any  $X, Y, Z \in \mathcal{T}_M^{1,0}$ ,*

$$\tilde{D}_X(\mathcal{E} \circ Y \circ Z) - \tilde{D}_Y(\mathcal{E} \circ X \circ Z) = \mathcal{E} \circ (\tilde{D}_X(Y \circ Z) - \tilde{D}_Y(X \circ Z)) \quad (58)$$

*ii) Assume that for any  $X, Y \in \mathcal{T}_M^{1,0}$ ,*

$$R_{X,Y}^{\tilde{D}} + [\tilde{C}_X, \tilde{C}_Y^b] = 0. \quad (59)$$

*Then the same relation holds with  $\tilde{D}$  replaced by  $D$ ,  $\tilde{C}$  replaced by  $C$  and  $\tilde{C}^b$  replaced by the adjoint  $C^b$  of  $C$  with respect to  $h$  if and only if, for any  $X, Y \in T^{1,0}M$ ,*

$$[\tilde{C}_X, \tilde{k}\tilde{C}_Y\tilde{k}] = [\tilde{C}_{\mathcal{E}^{-1} \circ X}, \tilde{k}\tilde{C}_{\mathcal{E}^{-1} \circ Y}\tilde{k}]. \quad (60)$$

*iii) If  $(T^{1,0}M, \tilde{C}, \tilde{h})$  is a harmonic Higgs bundle (respectively,  $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k})$  is a  $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$ -structure) then  $(T^{1,0}M, C, h)$  is a harmonic Higgs bundle (respectively,  $(T^{1,0}M, C, h, k)$  is a  $DChk$ -structure) if and only if both (58) and (60) are satisfied.*

*Proof.* We only need to check that (58) and (60) are equivalent to the  $tt^*$ -equations for  $(D, C, C^b)$ , the other statements being trivial from our previous considerations.

From a straightforward computation which uses  $\partial^{\tilde{D}}\tilde{C} = 0$ , for any  $X, Y \in \mathcal{T}_M^{1,0}$ ,

$$\begin{aligned} (\partial^D C)_{X,Y} &= \tilde{D}_X(\tilde{C}_{\mathcal{E}^{-1}})\tilde{C}_Y + \tilde{C}_{\mathcal{E}^{1/2}}\tilde{D}_X(\tilde{C}_{\mathcal{E}^{-1/2}})\tilde{C}_{\mathcal{E}^{-1} \circ Y} + \tilde{C}_{\mathcal{E}^{-1/2} \circ X}\tilde{D}_Y(\tilde{C}_{\mathcal{E}^{-1/2}}) \\ &\quad - \tilde{D}_Y(\tilde{C}_{\mathcal{E}^{-1}})\tilde{C}_X - \tilde{C}_{\mathcal{E}^{1/2}}\tilde{D}_Y(\tilde{C}_{\mathcal{E}^{-1/2}})\tilde{C}_{\mathcal{E}^{-1} \circ X} - \tilde{C}_{\mathcal{E}^{-1/2} \circ Y}\tilde{D}_X(\tilde{C}_{\mathcal{E}^{-1/2}}) \end{aligned}$$

To simplify notations, define  $T \in \text{End}_{\mathbb{C}}(T^{1,0}M)$  by  $T(X) := \mathcal{E}^{-1/2} \circ X$ . Therefore,  $\partial^D C = 0$  is equivalent with

$$\begin{aligned} &\left( T(\tilde{D}_X T) + (\tilde{D}_X T)T + T^{-1}(\tilde{D}_X T)T^2 \right) \tilde{C}_Y + T\tilde{C}_X\tilde{D}_Y T \\ &- \left( T(\tilde{D}_Y T) + (\tilde{D}_Y T)T + T^{-1}(\tilde{D}_Y T)T^2 \right) \tilde{C}_X - T\tilde{C}_Y\tilde{D}_X T = 0. \end{aligned}$$

On the other hand, applying the covariant derivative  $\tilde{D}_Y$  (for  $Y \in \mathcal{T}_M^{1,0}$ ) to the relation

$$T\tilde{C}_X = \tilde{C}_X T, \quad \forall X \in \mathcal{T}_M^{1,0}, \quad (61)$$

skew-symmetrizing in  $X$  and  $Y$  and using  $\partial^{\tilde{D}}\tilde{C} = 0$ , we obtain

$$(\tilde{D}_X T)\tilde{C}_Y - (\tilde{D}_Y T)\tilde{C}_X = \tilde{C}_Y(\tilde{D}_X T) - \tilde{C}_X(\tilde{D}_Y T). \quad (62)$$

Using (62), the condition  $\partial^D C = 0$  becomes equivalent to

$$\tilde{D}_X(T^2)\tilde{C}_Y = \tilde{D}_Y(T^2)\tilde{C}_X,$$

which, in turn, is equivalent to (58) (easy check). This proves claim *i*).

For claim *ii*), we need to prove that (60) is equivalent with the remaining  $tt^*$ -equation

$$R_{X,\bar{Y}}^D + [C_X, C_{\bar{Y}}^\flat] = 0, \quad \forall X, Y \in T^{1,0}M.$$

This follows from a straightforward computation which uses

$$R_{X,Y}^D = \tilde{C}_{\mathcal{E}^{1/2}} R_{X,Y}^{\tilde{D}} \tilde{C}_{\mathcal{E}^{-1/2}}, \quad \forall X, Y \in TM$$

together with

$$\tilde{C}_X^\flat = \tilde{k} \tilde{C}_{\bar{X}} \tilde{k}$$

and

$$C_X^\flat = k C_{\bar{X}} k = \tilde{C}_{\mathcal{E}^{1/2}} \tilde{k} \tilde{C}_{X \circ \mathcal{E}^{-1}} \tilde{k} \tilde{C}_{\mathcal{E}^{-1/2}}$$

for any  $X \in T^{0,1}M$ .  $\square$

We remark that condition (58) on the eventual identity is invariant under our duality of Theorem 3. The following simple result holds.

**Proposition 22.** *Let  $(M, *, \mathcal{E}, e)$  be the dual of  $(M, \circ, e, \mathcal{E})$ . If the eventual identity  $\mathcal{E}$  of  $(M, \circ, e)$  satisfies*

$$\tilde{D}_X(\mathcal{E} \circ Y \circ Z) - \tilde{D}_Y(\mathcal{E} \circ X \circ Z) = \mathcal{E} \circ \left( \tilde{D}_X(Y \circ Z) - \tilde{D}_Y(X \circ Z) \right), \quad (63)$$

*then the eventual identity  $e$  of  $(M, *, \mathcal{E})$  satisfies the dual condition*

$$D_X(e * Y * Z) - D_Y(e * X * Z) = e * (D_X(Y * Z) - D_Y(X * Z)), \quad (64)$$

*for any  $X, Y, Z \in \mathcal{T}_M^{1,0}$ .*

*Proof.* Straightforward computation, which uses (54) and (57).  $\square$

## 6.1 CV-structures and duality

A CV-structure on the holomorphic tangent bundle of a complex manifold  $M$  is a  $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$ -structure together with two endomorphisms  $\tilde{\mathcal{U}}$  and  $\tilde{\mathcal{Q}}$  of  $T^{1,0}M$ , satisfying some additional compatibility conditions. In particular, the endomorphism  $\tilde{\mathcal{Q}}$  is hermitian with respect to  $\tilde{h}$  and, as it turns out,  $\tilde{\mathcal{U}} = \tilde{C}_E$ , where  $E$  is an Euler vector field of weight one of the underlying  $F$ -manifold  $(M, \circ, e)$ .

It is immediately clear that CV-structures are not preserved by our duality of  $F$ -manifolds with eventual identities. The reason is that if  $E$  is an invertible Euler vector field on an  $F$ -manifold  $(M, \circ, e)$ , then  $e$  is not Euler for the dual  $F$ -manifold  $(M, *, E)$ . With this motivation, in Section 6.1.1 we define CV-structures in a weaker sense, with the Euler vector field replaced by an eventual identity. In Section 6.1.2 we prove that weak CV-structures so defined are preserved by our duality of  $F$ -manifolds with eventual identities, provided that the eventual identity satisfies conditions (58) and (60) of Theorem 21.

### 6.1.1 Weak CV-structures

We begin by recalling basic definitions and results about CV-structures on the holomorphic tangent bundle of a complex manifold. Our treatment of CV-structures follows closely [7], where more details and proofs can be found. It is worth remarking that sometimes our conventions differ from those used in [7]. While we use the generic notation  $\tilde{C}$  for a Higgs field and  $\tilde{C}^b$  for its adjoint with respect to a hermitian metric, the general notation in [7] for a Higgs field is  $C$  and  $\tilde{C}$  denotes its adjoint with respect to a hermitian metric. Moreover, in our conventions  $\tilde{C}$  is related to the associated multiplication  $\circ$  on the tangent bundle by  $\tilde{C}_X Y = X \circ Y$ , while in [7]  $C_X Y = -X \circ Y$ . Hopefully these differences will not generate any confusion.

**Definition 23.** A CV-structure is a  $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$ -structure  $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k})$  together with two endomorphisms  $\tilde{U}$  and  $\tilde{Q}$  of  $T^{1,0}M$  such that the following conditions hold:

- i) for any  $X \in T^{1,0}M$ ,  $[\tilde{C}_X, \tilde{U}] = 0$ .
- ii)  $\tilde{D}_{\tilde{X}}\tilde{U} = 0$  for any  $X \in T^{1,0}M$ , i.e. if  $Z \in \mathcal{T}_M$  then also  $\tilde{U}(Z) \in \mathcal{T}_M$ .
- iii) the  $(1,0)$ -part of  $\tilde{D}\tilde{U}$  has the following expression:
$$\tilde{D}_X\tilde{U} + [\tilde{C}_X, \tilde{Q}] - \tilde{C}_X = 0, \quad \forall X \in T^{1,0}M. \quad (65)$$
- iv)  $\tilde{Q}$  is hermitian with respect to  $\tilde{h}$ ; moreover,  $\tilde{Q} + \tilde{k}\tilde{Q}\tilde{k} = 0$ , or, equivalently,  $\tilde{Q}$  is skew-symmetric with respect to complex bilinear form  $\tilde{g}$  on  $T^{1,0}M$ , defined as usual by  $\tilde{g}(X, Y) = \tilde{h}(X, \tilde{k}Y)$ .

- v) the  $(1,0)$ -part of  $\tilde{D}\tilde{Q}$  has the following expression:

$$\tilde{D}_X\tilde{Q} - [\tilde{C}_X, \tilde{k}\tilde{U}\tilde{k}] = 0, \quad \forall X \in T^{1,0}M. \quad (66)$$

Let  $\circ$  be the multiplication on  $T^{1,0}M$ , related to the Higgs field  $\tilde{C}$  by  $X \circ Y := \tilde{C}_X Y$ , for any  $X, Y \in \mathcal{T}_M^{1,0}$  and denote by  $e \in \mathcal{T}_M$  its unity vector field. Recall that  $(M, \circ, e)$  is an  $F$ -manifold (this is a consequence of the  $tt^*$ -equation  $\partial^{\bar{\partial}}\tilde{C} = 0$ , see Lemma 4.3 of [7]). From i),  $\tilde{U}$  is the multiplication by a vector field  $\mathcal{E} = \tilde{U}(e) \in \mathcal{T}^{1,0}M$ . Condition ii) together with  $e \in \mathcal{T}_M$  imply that  $\mathcal{E}$  is holomorphic and condition (65) with  $\tilde{U} = \tilde{C}_{\mathcal{E}}$  implies that  $\mathcal{E}$  is an Euler vector field of weight one for  $(M, \circ, e)$  (again by Lemma 4.3 of [7]).

We now define the more general notion of weak CV-structures.

**Definition 24.** A weak CV-structure is a  $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$ -structure  $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k})$  together with two endomorphisms  $\tilde{U} = \tilde{C}_{\mathcal{E}}$  (where  $\mathcal{E} \in \mathcal{T}_M$ ) and  $\tilde{Q}$  of  $T^{1,0}M$ , satisfying all conditions of Definition 23, except that (65) is replaced by the weaker condition

$$\tilde{D}_X\tilde{U} + [\tilde{C}_X, \tilde{Q}] - \tilde{C}_{[\mathcal{E}, X]} = 0, \quad \forall X \in T^{1,0}M. \quad (67)$$

While a CV-structure determines a preferred Euler vector field on the underlying  $F$ -manifold, a weak CV-structure determines a vector field  $\mathcal{E}$  which satisfies the weaker condition (68), see below. In particular, if  $\mathcal{E}$  is invertible, then  $\mathcal{E}$  is an eventual identity.

**Lemma 25.** *Let  $(M, \circ, e)$  be an  $F$ -manifold and  $\tilde{D}$  a connection on  $T^{1,0}M$  such that  $\partial^{\tilde{D}}\tilde{C} = 0$ , where  $\tilde{C}_X Y = X \circ Y$  is the Higgs field. Let  $\mathcal{E}$  be a vector field of type  $(1, 0)$  on  $M$ .*

*i) Assume that*

$$L_{\mathcal{E}}(\circ)(X, Y) = [e, \mathcal{E}] \circ X \circ Y, \quad \forall X, Y \in \mathcal{T}_M^{1,0}. \quad (68)$$

*Then*

$$\tilde{D}_X(\tilde{C}_{\mathcal{E}}) + [\tilde{C}_X, \tilde{D}_{\mathcal{E}} - L_{\mathcal{E}}] - \tilde{C}_{[e, \mathcal{E}]} \tilde{C}_X = 0, \quad \forall X \in \mathcal{T}_M^{1,0}. \quad (69)$$

*ii) Conversely, assume that*

$$\tilde{D}_X(\tilde{C}_{\mathcal{E}}) + [\tilde{C}_X, \mathcal{Q}] - \tilde{C}_{[e, \mathcal{E}]} \tilde{C}_X = 0, \quad \forall X \in \mathcal{T}_M^{1,0}, \quad (70)$$

*for an endomorphism  $\tilde{Q}$  of  $T^{1,0}M$ . Then  $\mathcal{E}$  satisfies (68) and  $\tilde{Q}$  is equal to  $\tilde{D}_{\mathcal{E}} - L_{\mathcal{E}}$  up to addition with  $\tilde{C}_Z$ , for  $Z \in \mathcal{T}_M^{1,0}$ .*

*Proof.* Assume that (68) holds. Then, for any  $X \in \mathcal{T}_M^{1,0}$ ,

$$\begin{aligned} \tilde{D}_X(\tilde{C}_{\mathcal{E}}) + [\tilde{C}_X, \tilde{D}_{\mathcal{E}} - L_{\mathcal{E}}] &= \tilde{D}_X(\tilde{C}_{\mathcal{E}}) - \tilde{D}_{\mathcal{E}}(\tilde{C}_X) + [L_{\mathcal{E}}, \tilde{C}_X] \\ &= \tilde{C}_{[X, \mathcal{E}]} + L_{\mathcal{E}}(X \circ) = \tilde{C}_{[e, \mathcal{E}] \circ X}, \end{aligned}$$

where we used the  $tt^*$ -equation  $\partial^{\tilde{D}}\tilde{C} = 0$  and the condition (68). Our first claim follows. We now prove the second claim. As already mentioned above, if  $[e, \mathcal{E}] = e$  then (70) implies that  $\mathcal{E}$  is Euler of weight one. Without this additional assumption, the same argument shows that (70) implies (68). Therefore, (69) holds as well and  $\mathcal{Q} - \tilde{D}_{\mathcal{E}} + L_{\mathcal{E}}$  commutes with  $\tilde{C}_X$  for any  $X \in T^{1,0}M$ . Thus  $\mathcal{Q} - \tilde{D}_{\mathcal{E}} + L_{\mathcal{E}}$  is the multiplication by a vector field  $Z \in \mathcal{T}_M^{1,0}$ .  $\square$

The following Proposition provides a useful characterization of weak CV-structures. A similar statement for CDV-structures already appears in the literature (see Theorem 2.1 of [9]).

**Proposition 26.** *Let  $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k})$  be a  $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$ -structure. Define  $\tilde{g}(X, Y) = \tilde{h}(X, \tilde{k}Y)$  as usual and let  $\mathcal{E}$  be an eventual identity of the underlying  $F$ -manifold  $(M, \circ, e)$ . Then  $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k}, \tilde{\mathcal{U}} = \tilde{C}_{\mathcal{E}})$  extends to a weak CV-structure (i.e. there is an endomorphism  $\tilde{Q}$  of  $T^{1,0}M$  such that  $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k}, \tilde{\mathcal{U}}, \tilde{Q})$  is a weak CV-structure) if and only if there is  $Z \in \mathcal{T}_M$  such that*

$$L_{\mathcal{E}-\tilde{\mathcal{E}}}(\tilde{h})(X, Y) = \tilde{h}(X, Y \circ Z) - \tilde{h}(X \circ Z, Y), \quad \forall X, Y \in T^{1,0}M \quad (71)$$

*and*

$$L_{\mathcal{E}}(\tilde{g})(X, Y) = -2\tilde{g}(X \circ Y, Z), \quad \forall X, Y \in T^{1,0}M \quad (72)$$

*hold. Moreover,  $Z$  is uniquely determined by (72) and*

$$\tilde{Q} = \tilde{D}_{\mathcal{E}} - L_{\mathcal{E}} + \tilde{C}_Z. \quad (73)$$

*Proof.* Since  $\mathcal{E}$  is an eventual identity, Lemma 25 implies that any endomorphism  $\tilde{Q}$  such that  $(T^{1,0}M, \tilde{h}, \tilde{k}, \tilde{\mathcal{U}}, \tilde{Q})$  is a weak CV-structure must be of the form (73), with  $Z \in \mathcal{T}_M^{1,0}$ , and such that the relations

$$\tilde{h}(\tilde{Q}(Y), V) = \tilde{h}(Y, \tilde{Q}(V)) \quad (74)$$

and

$$\tilde{g}(\tilde{Q}(Y), V) + \tilde{g}(Y, \tilde{Q}(V)) = 0, \quad (75)$$

hold, for any  $Y, V \in \mathcal{T}_M^{1,0}$ . We will show that (71) and (72) are equivalent with (74) and (75) respectively. Since  $\tilde{D}$  is the Chern connection of  $(T^{1,0}M, \tilde{h})$ , for any  $X \in \mathcal{T}_M$  and  $Y, V \in \mathcal{T}_M^{1,0}$ ,

$$\begin{aligned} L_X(\tilde{h})(Y, V) &= X\tilde{h}(Y, V) - \tilde{h}(L_X Y, V) - \tilde{h}(Y, L_{\bar{X}} V) \\ &= \tilde{h}((\tilde{D}_X - L_X)(Y), V) - \tilde{h}(Y, (\tilde{D}_{\bar{X}} - L_{\bar{X}})(V)). \end{aligned}$$

On the other hand, since  $X$  is holomorphic and  $\tilde{D}^{(0,1)} = \bar{\partial}$ ,  $L_{\bar{X}} = \tilde{D}_{\bar{X}}$  on  $T^{1,0}M$  and we obtain

$$L_X(\tilde{h})(Y, V) = \tilde{h}((\tilde{D}_X - L_X)Y, V), \quad \forall Y, V \in \mathcal{T}_M^{1,0}. \quad (76)$$

Similarly,

$$L_{\bar{X}}(\tilde{h})(Y, V) = \tilde{h}(Y, (\tilde{D}_X - L_X)V), \quad \forall Y, V \in \mathcal{T}_M^{1,0}. \quad (77)$$

Relations (76) and (77) with  $X = \mathcal{E}$  imply that (71) is equivalent with (74). A similar argument which uses

$$L_X(\tilde{g})(Y, Z) = \tilde{g}((\tilde{D}_X - L_X)Y, Z) + \tilde{g}(Y, (\tilde{D}_X - L_X)Z) \quad (78)$$

shows that (72) is equivalent with (75).

Assume now that there is  $Z \in \mathcal{T}_M^{1,0}$  (uniquely determined, since  $\tilde{g}$  is non-degenerate) such that both (71) and (72) are satisfied and define an endomorphism  $\tilde{Q}$  of  $T^{1,0}M$  by (73). Then  $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k}, \tilde{\mathcal{U}}, \tilde{Q})$  is a weak CV-structure provided that relation (66) is satisfied. We will show that (66) is satisfied if and only if  $Z$  is holomorphic. For this we make the following computation: for any  $X \in \mathcal{T}_M$ ,

$$\tilde{D}_X(\tilde{D}_{\mathcal{E}} - L_{\mathcal{E}}) - [\tilde{C}_X, \tilde{k}\tilde{C}_{\mathcal{E}}\tilde{k}] = [\tilde{D}_X, \tilde{D}_{\mathcal{E}} - L_{\mathcal{E}}] + [\tilde{D}_X, \tilde{D}_{\mathcal{E}}] = \tilde{D}_{[X, \mathcal{E}]} - [\tilde{D}_X, L_{\mathcal{E}-\bar{\mathcal{E}}}] \quad (79)$$

where in the first equality we used

$$[\tilde{C}_X, \tilde{k}\tilde{C}_{\mathcal{E}}\tilde{k}] = -R_{X, \mathcal{E}}^{\tilde{D}} = -[\tilde{D}_X, \tilde{D}_{\mathcal{E}}] \quad (80)$$

(from the  $tt^*$ -equation and  $[X, \mathcal{E}] = 0$ ) and in the second equality we used  $\tilde{D}_{\mathcal{E}} = L_{\mathcal{E}}$ , because  $\mathcal{E} \in \mathcal{T}_M$ , and  $[\tilde{D}_X, \tilde{D}_{\mathcal{E}}] = \tilde{D}_{[X, \mathcal{E}]}$ , because the curvature of  $\tilde{D}$  is of type  $(1, 1)$ . On the other hand, using (71) and (72) and taking the Lie derivative of  $\tilde{g}(X, Y) = \tilde{h}(X, \tilde{k}Y)$  with respect to  $\mathcal{E}$ , we get

$$L_{\mathcal{E}-\bar{\mathcal{E}}}(\tilde{k}) = \tilde{k}\tilde{C}_Z + \tilde{C}_Z\tilde{k} \quad (81)$$

or, equivalently,

$$L_{\mathcal{E}-\tilde{\mathcal{E}}}(Y) = -\tilde{k}L_{\mathcal{E}-\tilde{\mathcal{E}}}(\tilde{k}Y) + \tilde{C}_Z Y + \tilde{k}\tilde{C}_Z\tilde{k}(Y), \quad \forall Y \in \mathcal{T}_M^{1,0}. \quad (82)$$

From (82), relation (79) becomes

$$\begin{aligned} \tilde{D}_X(\tilde{D}_{\mathcal{E}} - L_{\mathcal{E}}) - [\tilde{C}_X, \tilde{k}\tilde{C}_{\mathcal{E}}\tilde{k}] &= \tilde{D}_{[X,\mathcal{E}]} + [\tilde{k}\tilde{D}_{\tilde{X}}\tilde{k}, \tilde{k}L_{\mathcal{E}-\tilde{\mathcal{E}}}\tilde{k}] - [\tilde{D}_X, \tilde{C}_Z + \tilde{k}\tilde{C}_Z\tilde{k}] \\ &= \tilde{D}_{[X,\mathcal{E}]} + \tilde{k}[\tilde{D}_{\tilde{X}}, L_{\mathcal{E}-\tilde{\mathcal{E}}}] \tilde{k} - \tilde{D}_X(\tilde{C}_Z) - \tilde{k}\tilde{D}_{\tilde{X}}(\tilde{C}_Z)\tilde{k} \\ &= \tilde{D}_{[X,\mathcal{E}]} - \tilde{k}\tilde{D}_{[\tilde{X},\tilde{\mathcal{E}}]}\tilde{k} - \tilde{D}_X(\tilde{C}_Z) - \tilde{k}\tilde{D}_{\tilde{X}}(\tilde{C}_Z)\tilde{k} \\ &= -\tilde{D}_X(\tilde{C}_Z) - \tilde{k}\tilde{D}_{\tilde{X}}(\tilde{C}_Z)\tilde{k} \end{aligned}$$

where we used  $\tilde{D}_X(Y) = \tilde{k}\tilde{D}_{\tilde{X}}(\tilde{k}Y)$  (because  $\tilde{D}\tilde{k} = 0$ ) and

$$[\tilde{D}_{\tilde{X}}, L_{\mathcal{E}}] = [L_{\tilde{X}}, L_{\mathcal{E}}] = L_{[\tilde{X},\mathcal{E}]} = 0,$$

(because  $X, \mathcal{E} \in \mathcal{T}_M$ ). We deduce that

$$\tilde{D}_X(Q) - [\tilde{C}_X, \tilde{k}\tilde{C}_{\mathcal{E}}\tilde{k}] = -\tilde{k}\tilde{D}_{\tilde{X}}(\tilde{C}_Z)\tilde{k}.$$

Therefore, (66) is satisfied if and only if  $\tilde{D}_{\tilde{X}}(\tilde{C}_Z) = 0$ , for any  $X \in \mathcal{T}_M^{1,0}$ , i.e.  $Z$  is holomorphic. Our claim follows.  $\square$

### 6.1.2 Weak CV-structures and duality

Our aim in this Section is to prove the following result.

**Theorem 27.** *Let  $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k})$  be a  $\tilde{D}\tilde{C}\tilde{h}\tilde{k}$ -structure,  $\mathcal{E}$  an eventual identity on the underlying  $F$ -manifold  $(M, \circ, e)$  and  $\tilde{\mathcal{U}} := \tilde{C}_{\mathcal{E}}$ . Assume that conditions (58) and (60) are satisfied and let  $(T^{1,0}M, C, h, k)$  be the dual  $DChk$ -structure, as in Theorem 21. Then  $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k}, \tilde{\mathcal{U}})$  extends to a weak CV-structure if and only if  $(T^{1,0}M, C, h, k, \mathcal{U} := C_e)$  extends to a weak CV-structure.*

*Proof.* Assume that  $(T^{1,0}M, \tilde{C}, \tilde{h}, \tilde{k}, \tilde{\mathcal{U}})$  extends to a weak CV-structure. In order to apply Proposition 26 we need to determine an holomorphic vector field  $Z$  such that both (71) and (72) hold, with  $\circ$  replaced by  $*$ ,  $\tilde{h}$  replaced by  $h$  and  $\tilde{g}$  replaced by  $g$ . Define

$$Z := -(\tilde{D}_e e) \circ \mathcal{E} + \frac{1}{2}L_e(\mathcal{E}) \quad (83)$$

and notice that it is holomorphic: from the  $tt^*$ -equations and  $\tilde{D}^{(0,1)} = \bar{\partial}$ , we get:

$$\bar{\partial}_{\tilde{X}}(\tilde{D}_e e) = \tilde{D}_{\tilde{X}}\tilde{D}_e e = R_{\tilde{X},e}^{\tilde{D}}e = [\tilde{C}_e, \tilde{k}\tilde{C}_X\tilde{k}] = 0, \quad \forall X \in T^{1,0}M,$$

because  $e$  is holomorphic and  $\tilde{C}_e$  is the identity endomorphism. Therefore,  $\tilde{D}_e e$  and hence also  $Z$  is holomorphic. We now prove that the relations

$$L_e(g)(X, Y) = -2g(X * Z, Y), \quad \forall X, Y \in T^{1,0}M \quad (84)$$

and

$$L_{e-\bar{e}}(h)(X, Y) = h(X, Y * Z) - h(X * Z, Y), \quad \forall X, Y \in T^{1,0}M \quad (85)$$

hold, where  $*$  is the dual multiplication

$$X * Y = C_X Y = X \circ Y \circ \mathcal{E}^{-1}, \quad \forall X, Y \in T^{1,0}M. \quad (86)$$

Taking the Lie derivative with respect to  $e$  of the relation

$$g(X, Y) = \tilde{g}(X \circ \mathcal{E}^{-1}, Y)$$

and using (78) with  $X := e$ , together with  $L_e(\circ) = 0$  and

$$(\tilde{D}_e - L_e)(X) = (\tilde{D}_e e) \circ X, \quad \forall X \in T^{1,0}M \quad (87)$$

(relation (87) is an easy consequence of the  $tt^*$ -equation  $\partial \tilde{D} \tilde{C} = 0$ , for details see Theorem 4.5 of [7]), we get:

$$\begin{aligned} L_e(g)(X, Y) &= L_e(\tilde{g})(X \circ \mathcal{E}^{-1}, Y) + \tilde{g}(X \circ L_e(\mathcal{E}^{-1}), Y) \\ &= 2\tilde{g}((\tilde{D}_e e) \circ \mathcal{E}^{-1} \circ X, Y) + \tilde{g}(L_e(\mathcal{E}^{-1}) \circ X, Y) \\ &= 2g((\tilde{D}_e e) \circ X, Y) + g(\mathcal{E} \circ L_e(\mathcal{E}^{-1}) \circ X, Y) \\ &= -2g(X * Z, Y), \end{aligned}$$

for any  $X, Y \in T_M^{1,0}$ . Relation (84) follows. A similar computation shows that (85) holds as well. From Proposition 26,  $(T^{1,0}M, C, h, k, \mathcal{U})$  extends to a weak CV-structure.  $\square$

## 6.2 The semi-simple case

Recall that a holomorphic  $F$ -manifold  $(M, \circ)$  is called semi-simple if there are local coordinates  $(u^1, \dots, u^m)$  on  $M$  such that the multiplication  $\circ$  is diagonal (see Remark 7 iii). In the restricted case where the hermitian metric  $h$  and real structure  $\tilde{k}$  are also diagonal (and note that in general they need not be diagonal) the various conditions of Theorem 21 are automatically satisfied. More precisely, we can state.

**Example 28.** *Any eventual identity on a semi-simple  $F$ -manifold  $(M, \circ, \tilde{h}, \tilde{k})$  with hermitian metric and real structure taking the form*

$$\frac{\partial}{\partial u^i} \circ \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^j}, \quad \tilde{h}\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = H_{ii} \delta_{ij}, \quad \tilde{k}\left(\frac{\partial}{\partial u^i}\right) = k_i \frac{\partial}{\partial u^i},$$

(where  $|k_i| = 1$  and  $H_{ii} > 0$  for any  $i$ ) automatically satisfies the conditions (58) and (60).

*Proof.* We assume that the multiplication, hermitian metric and real structure are defined as above in canonical coordinates.

Let  $\mathcal{E}$  be an eventual identity, given by  $\mathcal{E} = \sum_{i=1}^n f_i \frac{\partial}{\partial u^i}$ . Recall that  $f_i$  depends on the variable  $u^i$  only. We will check (58) for fundamental vector fields  $X = \frac{\partial}{\partial u^i}$ ,  $Y = \frac{\partial}{\partial u^j}$  ( $i \neq j$ ) and  $Z = \frac{\partial}{\partial u^p}$ . Since the multiplication is semi-simple, (58) is clearly satisfied if  $p \notin \{i, j\}$ . If  $p = i$  say, then (58) becomes

$$\tilde{D}_{\frac{\partial}{\partial u^j}}(f_i \frac{\partial}{\partial u^i}) = \mathcal{E} \circ \tilde{D}_{\frac{\partial}{\partial u^j}}(\frac{\partial}{\partial u^i}),$$



or, since  $f_i$  depends only on  $u_i$  and  $i \neq j$ ,

$$f_i \tilde{D}_{\frac{\partial}{\partial u^j}} \left( \frac{\partial}{\partial u^i} \right) = \mathcal{E} \circ \tilde{D}_{\frac{\partial}{\partial u^j}} \left( \frac{\partial}{\partial u^i} \right). \quad (88)$$

On the other hand, since  $\tilde{D}$  is the Chern connection of  $\tilde{h}$ ,

$$\tilde{D}_X \left( \frac{\partial}{\partial u^i} \right) = \partial_X \log(H_{ii}) \frac{\partial}{\partial u^i}, \quad \forall X \in T^{1,0}M, \quad \forall i.$$

In particular,  $\tilde{D}_{\frac{\partial}{\partial u^j}} \left( \frac{\partial}{\partial u^i} \right)$  is a multiple of  $\frac{\partial}{\partial u^i}$  and (88) follows. We proved that relation (58) holds. It remains to prove relation (60). From the definitions of the real structure and multiplication in canonical coordinates, it can be checked that for any  $Y := \sum_{i=1}^n Y^i \frac{\partial}{\partial u^i}$ , the composition  $\tilde{k} \tilde{C}_Y \tilde{k}$  is the multiplication by the vector  $\sum_{i=1}^n \bar{Y}^i \frac{\partial}{\partial u^i}$ . In particular, both sides of (60) vanish. Our claim follows.  $\square$

It should be pointed out that the equations (58) and (60) place highly restrictive conditions on the various structures and may, in general, have no solution (as happens for some of the two-dimensional non-semi-simple examples in [17]). Just as almost-dual Frobenius manifolds satisfy almost all of the axioms of a Frobenius manifold, asking for the twisted structures to satisfy the full  $tt^*$  axioms may be too restrictive a condition. However, the above example does show that solutions in the semi-simple case - albeit in the subclass of diagonal real and hermitian structures - do exist.

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LIANA DAVID: Institute of Mathematics Simion Stoilow of the Romanian Academy, Calea Grivitei no. 21, Sector 1, Bucharest, Romania; E-mail address: liana.david@imar.ro

IAN A. B. STRACHAN: Department of Mathematics, University of Glasgow, Glasgow G12 8QW, UK; E-mail address: i.strachan@maths.gla.ac.uk